MOTIVIC NATURE OF CHARACTER VALUES OF DEPTH-ZERO REPRESENTATIONS

JULIA GORDON

ABSTRACT. In the present paper, it is shown that the values of Harish-Chandra distribution characters on definable compact subsets of the set of topologically unipotent elements of some reductive p-adic groups can be expressed as the trace of Frobenius action on certain geometric objects, namely, Chow motives. The result is restricted to a class of depth-zero representations of symplectic or special orthogonal groups that can be obtained by inflation from Deligne-Lusztig representations. The proof works both in positive and zero characteristic, and relies on arithmetic motivic integration.

1. INTRODUCTION

Our goal in this paper is to relate the distribution characters of depth-zero representations of p-adic groups with geometric objects, namely, Chow motives. This is part of an effort, initiated by T.C. Hales in [16], to express the concepts of representation theory of p-adic groups in such a way that would allow computations to be done without relying on the knowledge of the specific value of p (see also [17], [18] and [12]).

Let G be a p-adic group, for example, $G = \mathbf{G}(\mathbb{Q}_p)$ for a connected reductive algebraic group $\mathbf{G}$, and let π be a representation of G. Harish-Chandra [19] introduced the notion of the *distribution character* Θ_π of π and showed that it is represented by a locally integrable function θ_π on the group G. It turned out to be a difficult (and still unsolved in most cases) problem to give an explicit formula for the function θ_π. There is evidence [22], [15] suggesting that this difficulty is due to the existence of geometric objects that turn out to be *non-rational* whose number of points over the finite field $\mathbb{F}_p$ appears in the calculation of the value of the character. This demonstrates that harmonic analysis on reductive groups is, in general, *non-elementary* (see [15]), and the best we can hope for is not to obtain explicit formulas for character values, but to understand the underlying geometry. The first step towards this objective would be to establish the existence of geometric objects related to the values of characters in general. This is the step we carry out here for a class of depth-zero representations. The methods we use

are completely different from the ones in [15], and based on the new theory of *arithmetic motivic integration.*

In 1995, M. Kontsevich introduced the idea of motivic integration, which led J. Denef and F. Loeser to develop the theory of arithmetic motivic integration. This theory is outlined briefly in the Appendix. Arithmetic motivic integration allows to express the classical p-adic volumes of *definable* subsets of p-adic manifolds in terms of *trace of p-Frobenius action on virtual Chow motives* (which is, essentially, a generalization of counting $\mathbb{F}_p$-points of algebraic varieties). In [17], T.C. Hales outlined a program that applies arithmetic motivic integration in order to relate various quantities arising in representation theory of p-adic groups (such as orbital integrals) to geometry.

Here we use these ideas to express some averaged values of the character θ_π in terms of the Frobenius action on a Chow motive associated with the group, the representation, and the set of averaging. So far, such a result is restricted to the class of depth-zero supercuspidal representations of symplectic or special orthogonal (odd) groups, which can be obtained by inflation from Deligne-Lusztig representations. However, we expect that the methods should generalize to a much wider class, possibly including most supercuspidal representations.

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2. The statement

2.1. **Notation and assumptions.** Throughout the paper, we assume that the algebraic group $\mathbf{G}$ is a symplectic group $\mathbf{G} = \mathbf{Sp}(2n)$ or special orthogonal group $\mathbf{G} = \mathbf{SO}(2n+1)$ for some $n \in \mathbb{N}$. Specifically, we think of $\mathbf{Sp}(2n)$, resp., $\mathbf{SO}(2n+1)$ as the closed subgroup of $\mathbf{GL}(2n)$, resp., $\mathbf{GL}(2n+1)$, cut out by the condition ${}^t g J g = J$. The matrix $J = (J_{ij})$ of the size $2n$ (resp., $2n + 1$), in the symplectic case is defined by: $J_{ij} = 0$ if $i + j \neq 2n + 1$, $J_{i,2n+1-i} = 1$ if $i \leq n$, and -1 otherwise, and in the orthogonal case J is the anti-diagonal matrix with 1's on the anti-diagonal. The reason we have to restrict our attention only to these groups is explained in Section 5.6.

The letter 'F' will be reserved for a global field, and the letter 'E' – for a local field. The symbol $\mathcal{O}$ with various subscripts will always be used to denote the ring of integers of the corresponding field.

We will denote by $\mathcal{K}$ the collection of all nonarchimedean completions of a global field F. Namely, we consider two cases: if F is a number field, then $\mathcal{K} = \{F_v\}$ is the collection of its completions at nonarchimedean places. If

F is a function field, we denote by $\mathcal{K}$ the collection of all fields of the form $\mathbb{F}_x((t))$, where $x \in \operatorname{Spec} \mathcal{O}_F$ is a closed point.

If k is a finite field, and $\mathbf{G}$ – an algebraic group as above, we denote by $\mathbf{G}^k$ the finite group of k-points of $\mathbf{G}$, that is, the subgroup of $\mathbf{G}(\bar{k})$ consisting of the points fixed by the Frobenius action (see, e.g., [36, Chapter I] for details).

2.2. The representations.

Let $\mathcal{K}$ be a collection of local fields, as defined in Section 2.1. We would like to study the characters of representations of the groups $\mathbf{G}(E)$, as E ranges over the family $\mathcal{K}$. As stated in the introduction, the goal is to obtain an independent of E formula for the character. In particular, we need the representations to be, so to say, the "incarnations of the same representation" as we let E range over various completions of a given global field. To make this precise, we consider a family of representations of the groups $\mathbf{G}(E)$ (one representation for each group) that are parametrized by the same combinatorial data independent of E.

For the group $\mathbf{G}$, $\mathbf{G} = \mathbf{SO}(2n+1)$ or $\mathbf{G} = \mathbf{Sp}(2n)$ as above, the starting datum is a partition w of the integer n. For each finite field $\mathbb{F}_q$, w corresponds to a conjugacy class of elliptic tori in the finite group $\mathbf{G}^{\mathbb{F}_q}$ [39, Section 5.7]. In particular, for each field $E \in \mathcal{K}$ with ring of integers $\mathcal{O}_E$ and residue field k_E, w corresponds to a conjugacy class of tori in $\mathbf{G}^{k_E}$. We pick a torus $T = T_{k_E, w}$ from this class. Deligne-Lusztig theory associates a representation $R_{T,\chi}^{\mathbf{G}^{k_E}}$ with the data (T, χ) where χ is a character of the torus T. We let the character χ be an arbitrary irreducible character of T in general position. Such character exists if the characteristic of k_E is large enough; it will be explained in Section 2.4 why the choice of χ subject to these restrictions does not matter. Let $\psi = R_{T,\chi}^{\mathbf{G}^{k_E}}$ be the Deligne-Lusztig representation of $\mathbf{G}^{k_E}$ corresponding to the pair (T, χ) (see Appendix, Section 5.6). It is a representation of $\mathbf{G}^{k_E}$ on a $\bar{\mathbb{Q}}_l$-vector space with $l \neq \operatorname{char} k_E$, but its construction does not depend either on k_E or on l. We would like to be able to vary E (and with it, k_E). Therefore, let us fix, once and for all, $l = 5$ (see Section 5.6 for the explanation of this choice), and only consider the local fields in $\mathcal{K}$ with the residual characteristic greater than 5 from now on. We also embed $\bar{\mathbb{Q}}_5$ in $\mathbb{C}$, and replace ψ with the representation $\bar{\psi}$ on the resulting complex vector space.

The representation $\pi_{E,w,\chi}$ of the group $\mathbf{G}(E)$ associated with the partition w and the character χ is constructed as follows. Let $K_E = \mathbf{G}(\mathcal{O}_E)$. The group K_E is a maximal compact subgroup of $\mathbf{G}(E)$. First, we inflate the representation $\bar{\psi}$ to a representation of K_E, and then induce the resulting representation from K_E to $G(E)$ (by means of compact induction). By 'inflation' we mean the following process. Denote by $\operatorname{Res} : K_E \to \mathbf{G}^{k_E}$ the map that acts, coordinate-wise, as reduction modulo the uniformizer of the field E. Then define the representation $\kappa : K_E \to \operatorname{End}(V)$ by $\kappa(g) := \bar{\psi}(\operatorname{Res}(g))$, where V is the representation space of $\bar{\psi}$. The final result of

this inflation-induction procedure is a depth-zero representation $\pi_{E,w,\chi} =$ c-$\mathrm{Ind}_{K_E}^{\mathbf{G}(E)}\kappa$ of the group $\mathbf{G}(E)$ on a complex vector space.

It is relevant to note here that the construction so far is not quite independent of the field $E \in \mathcal{K}$, because of the presence of χ. However, later we will see that χ drops out of all calculations of the character values. For now, it may be more precise to talk about a "pack" of representations $\pi_{E,w,\chi}$ associated with the data (E,w).

The following known fact is important for us, so we state it as a lemma, but do not include the proof (cf. [27, Proposition 6.8]).

Lemma 1. *The representation $\pi_{E,w,\chi}$ is supercuspidal.*

2.3. **The main theorem.** Let $\mathbf{G}$, $\mathcal{K}$, w, χ, and $\pi_{E,w,\chi}$ for almost all $E \in \mathcal{K}$ be as in Section 2.2. For almost all $E \in \mathcal{K}$ the residual characteristic of E is large enough so that the construction of the Section 2.2 makes sense.

We denote the distribution character of the representation $\pi = \pi_{E,w,\chi}$ by $\Theta_{w,E}$, even though it would be more precise to denote it by $\Theta_{w,\chi,E}$. We are omitting χ from the notation because our calculations do not depend on χ. We denote the locally summable function that represents the distribution $\Theta_{w,E}$ by $\theta_{w,E}$, and also call it the character of π.

Let K_E^{rtu} denote the set of regular topologically unipotent elements in $K_E = \mathbf{G}(\mathcal{O}_E)$. Denote by I_E the Iwahori subgroup of the group $\mathbf{G}(E)$ which consists of the elements in K_E whose reduction modulo the uniformizer is an upper-triangular matrix in the standard representation.

To state the main theorem, we need two notions: of Pas's language, and of virtual Chow motives. They are defined in the Appendix: Sections 5.1, 5.2, and 5.4. The completed ring of virtual Chow motives which "arise from varieties" (see Apenndix, Section 5.4) is denoted by $\hat{K}_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$.

Let α be a formula in Pas's language (see Appendix, Section 5.2) defining a compact subset $\Gamma_{\alpha,E} = Z(\alpha,E) \subset K_E^{rtu}$ for almost all $E \in \mathcal{K}$. Let *vol* stand for the p-adic Haar measure on $G = \mathbf{G}(E)$ normalized so that its restriction to K_E coincides with the Serre-Oesterlé measure (see Appendix, Section 5.5).

Theorem 2. *Given $\mathbf{G}$, w and $\mathcal{K}$ as above, there exists a virtual Chow motive $M_{\alpha,w} \in \hat{K}_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$, such that for almost all $E \in \mathcal{K}$, the following equality holds:*

$$vol(I_E)\Theta_{w,E}(\Gamma_{\alpha,E}) = vol(I_E)\int_{\Gamma_{\alpha,E}} \theta_{w,E}(\gamma)\,d\gamma = TrFrob_E(M_{\alpha,w}),$$

where $Frob_E$ is the Frobenius action corresponding to the place of F that gave rise to the field E, and $\Theta_{w,E}(\Gamma_{\alpha,E})$ stands for the value of the distribution $\Theta_{w,E}$ at the characteristic function of the set $\Gamma_{\alpha,E}$.

As announced in the introduction, this theorem states that the value of the distribution character on a *definable* compact subset of K^{rtu} can be recovered from a geometric object (namely, a virtual Chow motive), up to

a factor which is a polynomial in the cardinality of the residue field (the volume of the Iwahori subgroup).

2.4. **Remarks. 1.** A new version of the theory of motivic integration, that is about to appear, yields that the ring of virtual Chow motives does not need to be completed in order to define the elements $M_{\alpha,w}$ (see Appendix, Section 5.4).

2. The reason we restrict our attention only to the topologically unipotent elements is the following. Recall that the irreducible character χ of the torus in the finite group $\mathbf{G}^{k_E}$ is part of the construction of the representation π which does not have a nice combinatorial parametrization. However, as we will see in the next section, the value of $\theta_{w,E}$ at a *topologically unipotent* element is expressed through the values of the character of Deligne-Lusztig representation $\bar{\psi}$ that gave rise to π at *unipotent* conjugacy classes. Those values, in turn, are expressed by Green polynomials (see Appendix, Section 5.6, for a brief review and references). We are using the fact that the values of the Green polynomials at unipotent elements do not depend on the choice of the character χ.

3. We are considering the averages of the function $\theta_{w,E}$ over some defin-able subsets of K_E^{rtu}. The other way of saying this is to say that we look at the values of the distribution character $\Theta_{w,E}$ itself at the characteristic functions of those subsets. In fact, we would like to study the individual values $\theta_{w,E}(\gamma)$ at regular elements. Pas's language, which is one of our main tools, does not allow to handle individual elements of p-adic fields or p-adic groups. As a function on the set of regular elements in the p-adic group $\mathbf{G}(E)$, the character $\theta_{w,E}(\gamma)$ is locally constant. However, there are no explicit results that say "how small" the sets on which $\theta_{w,E}$ is constant might be. This forces us to average $\theta_{w,E}$ over some sets that we can control. However, the actual shape of the sets of averaging is flexible (the only requirement being that they are *definable*, see Section 5.2).

The rest of the paper is devoted to the proof of Theorem 2. In the next section we do an entirely p-adic calculation which serves as a preparation for the proof, and in Section 4 we give it a motivic interpretation, which leads to the desired result.

3. The Character: a p-adic calculation

Throughout this section we adhere to a fixed local field E with a uni-formizer ϖ, ring of integers $\mathcal{O}$, and residue field $k = \mathcal{O}/(\varpi)$. Since the field E stays fixed here, we drop the subscript E from all notation for simplicity.

3.1. **Character of an induced representation.** Let π be a representation of $G = \mathbf{G}(E)$ obtained from a Deligne-Lusztig representation $\bar{\psi}$ of $\mathbf{G}^k$ by the inflation and induction procedure described in Section 2.2. By Lemma 1, π is supercuspidal.

We fix $K := K_E$ – a maximal compact subgroup of G, and let $I = I_E$.

We denote by ρ the character of the representation $\bar{\psi}$, and by $\tilde{\rho}\colon K \to \mathbb{C}$ – the character of its inflation to K. Note that the value $\tilde{\rho}(g)$ at $g \in K$ depends only on the reduction of g modulo ϖ.

Let Γ be a compact subset of K^{rtu}. By the Frobenius-type formula for induced character, [31, Theorem 1.9] and [1, Theorem A.14(ii)] (see also the Remark at the end of the Appendix in [1]), the value of the character $\theta_w(\gamma)$ can be expressed as a sum that is *finite* uniformly for all $\gamma \in \Gamma$:

$$\text{(1)} \qquad \theta_w(\gamma) = \sum_{a \in I \backslash G / K} \left(\sum_{y \in IaK/K} \tilde{\rho}(y^{-1}\gamma y) \right).$$

Note that in [31], the double cosets are taken with respect to the same subgroup on the left and right; however, the modification we are using here can be obtained in the exactly same way. Our version of the formula coincides with the formula in [1].

3.2. **The double cosets.** The indexing set $I \backslash G / K$ in the outer sum in (1) can be described using a version of the Iwasawa decomposition.

Let A be the split standard torus in G. We can think of the elements of A as diagonal matrices of the form $\operatorname{diag}(u_i \varpi^{\lambda_i})_{i=1,\dots,r}$ with $u_i \in \mathcal{O}^*$ and $\lambda_i \in \mathbb{Z}$ satisfying the relations forced by the definition of the group (for example, in the symplectic case, $u_i = u_{r-i}^{-1}$ and $\lambda_i = -\lambda_{r-i}$). Then, by a version of Iwasawa decomposition, $G = IAK$ [21].

Observe that the intersection of the subgroups I and A is exactly the intersection of A with K. It follows that every left coset of G modulo K has a representative of the form ya with $y \in I$ and $a \in A_0$, where A_0 is the set of elements in A of the form $a_\lambda = \operatorname{diag}(\varpi^{\lambda_i})_{i=1,\dots,r}$, where $\lambda = (\lambda_i)_{i=1,\dots,r} \in \mathbb{Z}^r$ satisfies the conditions mentioned above. In particular, the set of double cosets $I \backslash G / K$ is in bijection with the set A_0.

Fix a multi-index $(\lambda_i)_{i=1,\dots,r}$ that gives an element $a_\lambda \in A_0$.

Definition 3. We call the elements $y_1, y_2 \in I$ λ-equivalent if $y_1 a_\lambda$ and $y_2 a_\lambda$ belong to the same left K-coset (that is, $a_\lambda^{-1} y_1^{-1} y_2 a_\lambda \in K$). We denote the λ-equivalence class of $y \in I$ by $[y]_\lambda$.

As an element of the torus A, a_λ may be considered as a lift of an element of the extended Weyl group of G. Let l_λ be the length of this element. By a theorem of Iwahori and Matsumoto [21], the set of λ-equivalence classes is in bijection with $\mathbb{A}^{l_\lambda}(k)$.

The following two simple observations will be used below, so we state them as a lemma. We keep the same notation as above, and let $q = |\mathcal{O}/(\varpi)|$ denote the cardinality of the residue field.

Lemma 4. 1. *If y_1 and y_2 are λ-equivalent, then the elements $(y_1 a_\lambda)^{-1}\gamma(y_1 a_\lambda)$ and $(y_2 a_\lambda)^{-1}\gamma(y_2 a_\lambda)$ are conjugate by an element of K.*
2. *All the λ-equivalence classes have equal volumes, equal to $\frac{vol(I)}{q^{l_\lambda}}$.*

Proof. The first statement is obvious: $(y_1 a_\lambda)^{-1} \gamma (y_1 a_\lambda)$
$= k(y_2 a_\lambda)^{-1} \gamma (y_2 a_\lambda) k^{-1}$, where $k = a_\lambda^{-1} y_1^{-1} y_2 a_\lambda \in K$ by definition of λ-equivalence.

To show the second assertion, note that y and y_0 are λ-equivalent if and only if $y^{-1} \in a_\lambda K a_\lambda^{-1} y_0^{-1} \cap I$. Hence the volume of each equivalence class equals the volume of the set $I \cap a_\lambda K a_\lambda^{-1}$ (since the group G is reductive, it is unimodular, and therefore the operation of taking an inverse preserves the Haar measure). Let us compute this volume. The total number of equivalence classes within the Iwahori subgroup I equals $q^{l\lambda}$. Hence, the volume of each equivalence class is $\frac{vol(I)}{q^{l\lambda}}$. $\qquad \square$

3.3. **A formula for the character.** First, let us introduce some notation. Let C be a conjugacy class in the finite group $\mathbf{G}^k$, and let $\gamma \in K^{rtu}$. For $\lambda \in \mathbb{Z}^r$ as in the previous section, denote by $N_C^\lambda(\gamma)$ the number of λ-equivalence classes $[y]_\lambda$ of elements of I such that the following conditions are satisfied:

  i) $(ya)^{-1} \gamma (ya) \in K$ for any $y \in [y]_\lambda$
  ii) $\text{Res}((ya)^{-1} \gamma (ya)) \in C$ for any $y \in [y]_\lambda$.

The condition (i) is well defined by Lemma 4. Let $N_C(\gamma) = \sum_\lambda N_C^\lambda(\gamma)$. We observe that for each element $\gamma \in K^{rtu}$ the sum is finite, because the set of classes $[y]_\lambda$ satisfying the condition (i) is nonempty only for a finite set of multi-indices λ, by [1, Theorem A.14(ii)].

Using the fact that $\tilde{\rho}$ is a lift of the character ρ of the finite group $\mathbf{G}^k$, the sum (1) can be rewritten as a sum over the conjugacy classes C of $\mathbf{G}^k$:

$$(2) \qquad \theta_w(\gamma) = \sum_C \sum_{\substack{(\lambda, [y]_\lambda) \\ \forall y \in [y]_\lambda (ya)^{-1}\gamma(ya) \in K, \\ \text{Res}((ya)^{-1}\gamma(ya)) \in C}} \rho(C) = \sum_C N_C(\gamma)\rho(C),$$

where C runs over the conjugacy classes of $\mathbf{G}^k$.

Lemma 5. 1. *If Γ is a compact subset of the set of regular elements in G, then there exists a finite set A_Γ of indices λ so that $N_C^\lambda(\gamma) = 0$ for all $\gamma \in \Gamma$ and $\lambda \notin A_\Gamma$.*
2. *If γ is topologically unipotent, only the unipotent conjugacy classes C appear in the summation in (1).*

Proof. The first statement is part of the statement of [1, Theorem A.14(ii)]. (In [1], the Theorem is stated only for GL_n and its relatives, but as the authors point out at the end of the Appendix, the argument carries over to classical groups without any changes.)

The second statement is based on the following two simple observations. First, the eigenvalues of $\text{Res}(\gamma)$ are obtained from the eigenvalues of γ by reduction modulo the uniformizer of the valuation extended to the algebraic closure of the field E. Second, it follows, for example, from [2, Lemma 2.2] that the eigenvalues of a topologically unipotent element (in the algebraic closure of the given local field) are congruent to 1. $\qquad \square$

3.4. **Averages.** Let Γ be a compact subset of $K = \mathbf{G}(\mathcal{O})$. The goal is to express the average $\Theta_w(\Gamma)$ of the values of θ_w over the set Γ as the p-adic volume of some p-adic object. In the next section we will use Denef and Loeser's "comparison theorem" to relate the resulting p-adic volume to a Chow motive. We do it by means of artificially constructing some subsets of $G \times G$ whose p-adic volumes equal the numbers $N_C(\gamma)$ that appear in formula (2), and showing that these subsets are definable.

Consider the Cartesian product

$$\mathcal{X} = \mathbf{G} \times_{\mathcal{O}} \mathbf{G}.$$

Fix an element $a = a_\lambda$ as above and consider the double coset $D_a = IaK \subset G$. Let $W_\lambda^E(\Gamma)$ be the subset of $\mathcal{X}(\mathcal{O})$ defined by

$$W_\lambda^E(\Gamma) = \{(y, \gamma) \mid y \in I, \gamma \in \Gamma, a^{-1}y^{-1}\gamma ya \in K\}.$$

The set $W_\lambda^E(\Gamma)$ is partitioned into subsets

$$W_{C,\lambda}^E(\Gamma) = \{(y, \gamma) \in W_\lambda^E(\Gamma) \mid \mathrm{Res}((ya)^{-1}\gamma(ya)) \in C\},$$

where C runs over the unipotent conjugacy classes of $\mathbf{G}^k$.

Let $W_{C,\lambda}^E(\gamma) \subset I$ be the projection onto the first coordinate of the cross-section of $W_{C,\lambda}^E(\Gamma)$ with fixed $\gamma \in \Gamma$.

Let vol be the Haar measure on $\mathbf{G}(E)$ normalized so that its restriction to K coincides with the Serre-Oesterlé measure. Then the space $\mathcal{X}(\mathcal{O})$ has the natural product measure.

Main Observation. Up to a normalization factor, the number $N_C^\lambda(\gamma)$ defined in Section 3.3 is the volume of the set $W_{C,\lambda}^E(\gamma)$:

$$N_C^\lambda(\gamma) = \frac{q^{l_\lambda}}{vol(I)} vol(W_{C,\lambda}^E(\gamma)),$$

where q is the cardinality of the residue field k, l_λ is an integer that depends only on λ, and I is the Iwahori subgroup.

Proof. Fix a multi-index λ. By definition, $N_C^\lambda(\gamma)$ is the number of equivalence classes $[y]_\lambda$ of the elements $y \in I$ that satisfy the conditions (i) and (ii). The set $W_{C,\lambda}^E(\gamma)$ is a disjoint union of these equivalence classes. Recall that by Lemma 3.2, all λ-equivalence classes have equal volumes, equal to $\frac{vol(I)}{q_\lambda^l}$. Hence,

$$N_C^\lambda(\gamma) = \frac{q^{l_\lambda}}{vol(I)} vol(W_{C,\lambda}^E(\gamma)).$$

In the next section, we will also need the following property of the normalization factor that appears in the above formula.

Remark 6. The factor $p(q) = \frac{q^{l_\lambda}}{vol(I)}$, as a function of q, is an element of $\mathbb{Z}(x)$, since the volume of I is a polynomial in q with coefficients in $\mathbb{Z}$ that depends only on the group $\mathbf{G}$.

Consider the average (with respect to the Haar measure *vol* defined above) of the values of θ_w over the set Γ. In order to get rid of denominators, we multiply it by $vol(I)$.

(3)
$$vol(I)\Theta_w(\Gamma) = vol(I)\int_\Gamma \theta_w(\gamma)\,d\gamma = vol(I)\int_\Gamma \sum_C \sum_\lambda N_C^\lambda(\gamma)\rho(C)$$

$$(4) \qquad = \sum_C \sum_\lambda \rho(C)q^{l_\lambda}\int_\Gamma vol(W_{C,\lambda}^E(\gamma))\,d\gamma$$

$$(5) \qquad = \sum_C \sum_\lambda \rho(C)q^{l_\lambda}\iint_{W_{C,\lambda}^E(\Gamma)} 1\,dyd\gamma = \sum_C \sum_\lambda \rho(C)q^{l_\lambda}vol(W_{C,\lambda}^E(\Gamma)).$$

Note that the sum and the integral in the expression (4) can be interchanged because the summation over λ, in fact, runs over a finite set A_Γ, by Lemma 5. This equality is the basis for our main result, as the next section shows.

4. Motivic interpretation

4.1. **Varying the field.** In this section, we complete the proof of Theorem 2 by associating virtual Chow motives with the subsets $W_{C,\lambda}^E(\Gamma)$ from the previous section using arithmetic motivic integration. This leads to an expression of the average value of θ_w as a trace of Frobenius on a certain virtual Chow motive.

Throughout this section, we work with the notation and all the assumptions of Theorem 2, so that $\mathcal{K}$ is the collection of all nonarchimedean completions of a given global field F; $\mathbf{G}$, π, w are as in Section 2.2, α is a formula in Pas's language, and for $E \in \mathcal{K}$, $\Gamma_{\alpha,E}$ is a definable subset of K_E defined by the formula α. As in Theorem 2, we are assuming that $\Gamma_{\alpha,E}$ is contained in the set of regular topologically unipotent elements in K_E. In order to justify the assumption that a *definable* set $\Gamma_{\alpha,E}$ can be contained in K^{rtu}, we would like to show that the set K^{rtu} itself is definable.

Lemma 7. *Let $E \in \mathcal{K}$ be a local field. Let $K = K_E$. Then*
1. *The set K^{rtu} is definable.*
2. *The Iwahori subgroup I_E is a definable subset of K_E.*

Proof. 1. Let γ be an element of K^{rtu}. We use the standard representation of the group G to think of γ as a matrix $\gamma = (x_{ij})$, and let the 'x_{ij}' be the free variables in all our logical formulas. We use the following definition of *regular* [24]. Let n be the dimension of $\mathbf{G}$ and l be the rank of $\mathbf{G}$. Then the element γ is regular if and only if $D_l(\gamma) \neq 0$, where $D_l(\gamma)$ is defined by the following expression:

$$\det((t+1)I_n - \mathrm{Ad}(\gamma)) = t^n + \sum_{j=0}^{n-1} D_j(\gamma)t^j.$$

We observe that $\mathrm{Ad}(\gamma)$ can be thought of as a matrix expression whose entries are polynomials in x_{ij}, and hence $D_l(\gamma)$ is also a polynomial expression in the variables x_{ij}. Therefore, '$D_l(\gamma) \neq 0$' is a formula in Pas's language. In conjunction with the formulas coming from the equations of the group $\mathbf{G}$ as an affine variety, and with the Pas's language formulas '$\mathrm{ord}(x_{ij}) \geq 0$' for each index (i, j), this formula defines the set of regular elements in K.

To cut out the subset of topologically unipotent elements, we use Lemma 5, which implies that γ is topologically unipotent if and only if the element $\mathrm{Res}(\gamma)$ is unipotent, and then use the combinatorial (independent of the field) parametrization of the set of unipotent conjugacy classes in the finite group $\mathbf{G}^k$ (where k is the residue field of E). The details are part of the proof of Lemma 9 below.

2. The Iwahori subgroup I that we are considering is defined by the formulas '$\mathrm{ord}(x_{ij}) \geq 0$', $i \leq j$, in conjunction with the formulas '$\mathrm{ord}(x_{ij}) \geq 1$' for all pairs of indices (i, j) with $i > j$. $\qquad\square$

4.2. Preparation: the subsets $W_{C,\lambda}^E(\Gamma_{\alpha,E})$ are definable.

We start by fixing a multi-index λ, and considering the corresponding double coset D_a, as in Section 3.4. Let $l = l_\lambda$. Recall that α stands for a formula in Pas's language that defines a compact subset $\Gamma_{\alpha,E}$ of K_E^{rtu} for almost all $E \in K$. Since we are keeping α fixed, in this section we drop '$\Gamma_{\alpha,E}$' from the notation, and denote the sets whose definability we wish to prove simply by $W_{C,\lambda}^E$.

Lemma 8. *Let λ be a fixed index. Under the assumptions of Theorem 2, $W_\lambda^E(\Gamma_{\alpha,E})$ is a definable subset of $\mathbf{G} \times \mathbf{G}(E)$ for $E \in \mathcal{K}$.*

Proof. Suppose for a moment that the field $E \in \mathcal{K}$ is fixed, an element $y = (y_{ij}) \in I_E$ is fixed, and take $\gamma = (\gamma_{ij}) \in \Gamma_{\alpha,E}$. We observe that each matrix coefficient of the matrix $y^{-1}\gamma y$ is a polynomial, with coefficients in $\mathbb{Z}$, homogeneous in the variables y_{ij} and linear in the variables γ_{ij}. We denote these polynomials by $P_{\kappa\eta}(y_{ij}, \gamma_{ij})$, $\kappa, \eta = 1, \ldots, r$. The conjugation by a_λ multiplies each entry of $y^{-1}\gamma y$ by an integral power of the uniformizer that depends only on the multi-index λ. We denote these powers by $n_{\kappa\eta}$.

Consider the formulas

(6)
$$\text{'}\mathrm{ord}P_{\kappa\eta}(y_{ij}, \gamma_{ij}) + n_{\kappa\eta} \geq 0\text{'}, \quad \text{'}\mathrm{ord}(y_{ij}) \geq 0\text{'}, i \leq j, \quad \text{'}\mathrm{ord}(y_{ij}) \geq 1\text{'}, i > j.$$

Let ϕ be the conjunction of all the formulas (6) and the formula α in the variables γ_{ij} that defines $\Gamma_{\alpha,E}$. Then $W_\lambda^E(\Gamma_{\alpha,E}) = Z(\phi, E)$ in the notation of Section 5.2 of the Appendix. $\qquad\square$

Lemma 9. *The subsets $W_{C,\lambda}^E$ are definable.*

Proof. Suppose for a moment that a local field $E \in \mathcal{K}$ is fixed. All we need to show is that the set of topologically unipotent elements $y \in K$ such that $\mathrm{Res}(y) \in C$ is definable for each conjugacy class C of the group $\mathbf{G}^k$ that appears in the summation (3). In order to write a formula that cuts out this set, we let the matrix coefficients of y be the free variables, as before.

First of all, we observe that the symbol 'Res' can be used in formulas in Pas's language with the following meaning: if $\varphi(y)$ is an expression in Pas's language with y – a variable of the residue field sort, then we set $\varphi(\mathrm{Res}(x))$ (where x is a free variable of the valued field sort) to be $\varphi_1 \vee \varphi_2$, where $\varphi_1 :=$'$\mathrm{ord}(x) = 0$'$\wedge\varphi(\overline{\mathrm{ac}}(x))$, and $\varphi_2 :=$'$\mathrm{ord}(x) > 0$'$\wedge\varphi(0)$. Also, if x is an abbreviation for a matrix or a vector of free variables, we understand '$\mathrm{Res}(x)$' in a natural way, as the application of the symbol 'Res' to each component of x, as described above.

Since $\Gamma_{\alpha,E}$, by assumption, is contained in the set of topologically unipotent elements in G, only the *unipotent* conjugacy classes C appear in (3), by Lemma 5. As described in the Appendix, Section 5.7, the set of unipotent conjugacy classes C is in bijection with the set of purely combinatorial data $\{(\Lambda, (\epsilon_i))\}$, where Λ is a partition of r, and $\epsilon_i \in k_E^* / k_E^{*2}$ (where k_E is the residue field of E). Suppose that the class C corresponds to a pair $(\Lambda, (\epsilon_i))$. We think of (ϵ_i) as a sequence with entries 0 or 1.

In order to write down the logical formula $\phi_{\Lambda,(\epsilon_i)}$ cutting out the set of elements whose reductions fall into the conjugacy class C, we need to unwind the process described in Section 5.7 that associates C with the data $(\Lambda, (\epsilon_i))$.

First, for $y \in K$, y – topologically unipotent, let $Y = (1 - y)(1 + y)^{-1}$ be the Cayley transform of y (note that the matrix $1 + y$ is invertible, since y is assumed to be topologically unipotent). The element Y lies in the Lie algebra of $\mathbf{G}$ over the given p-adic field. Let $\tilde{Y}_1 = \det(1 + y)Y$, and let '$Y_1$'='$\mathrm{Res}(\tilde{Y}_1)$' (so that '$Y_1$' in all formulas is treated as a matrix with components ranging over the residue field sort, even though, to be precise, all the formulas involving 'Y_1' would be conjunctions of formulas with free variables ranging over the valued field sort, namely, the matrix coefficients of $\tilde{Y}_1$). Note that the matrix coefficients of $\tilde{Y}_1$ are polynomials, with $\mathbb{Z}$-coefficients, in the matrix coefficients of y. Hence, finally, the abbreviation 'Y_1' can be used instead of 'y' in all subsequent formulas, and we treat it just as a matrix with components of the residue filed sort.

The first part of the data, Λ, prescribes the set of Jordan blocks of the matrix Y, which is the same as the set of Jordan blocks of Y_1. The set of all elements Y_1 whose Jordan blocks match the partition Λ is defined by

$$(7) \qquad\qquad `\exists (g_{ij})_{i,j=1,\ldots,r} : (g_{ij})Y_1(g_{ij})^{-1} = J_\Lambda`,$$

where J_Λ is the matrix with entries 0 and 1, consisting of Jordan blocks given by the partition Λ. Note, again, that in fact formula (7) has only free variables of the valued field sort, that is, the matrix coefficients 'y_{ij}' (we see that by unwinding the abbreviation 'Y_1'). All the variables ranging over the residue field sort (that is, 'g_{ij}') are bound. Hence, logical formula (7) cuts out a definable set.

Second, we need to cut out the subset of this definable set that corresponds to the given sequence (ϵ_i). Recall (from Section 5.7) that the sequence (ϵ_i) comes from a collection of quadratic forms on the vector spaces V_i. Let

$c_i = \dim V_i$. We will need to introduce separately a few components of the final formula that would define $W_{C,\lambda}^E$.

Let us look at one entry ϵ_i corresponding to the vector space (V_i, q_i).

Let '$\mathrm{li}(v_1, \ldots, v_m)$' be the logical formula with the free variables $v_{11}, \ldots, v_{mr}$ that states that the vectors $v_1 = {}^t(v_{11}, \ldots, v_{1r}), \ldots, v_m = {}^t(v_{m1}, \ldots, v_{mr})$ are linearly independent (here $r = \dim V$), see [12].

Let '$\ker_A(v)$' be the logical formula stating that the vector v is annihilated by the linear operator A, where 'A' is a matrix of logical terms. The free variables of this formula are the components of v and the matrix coefficients of A. Let '$\mathrm{q\text{-}span}(Y_1, v_1, \ldots, v_m)$' be the formula with free variables $v_1, \ldots, v_m$ and Y_1 (each of which is an abbreviation for a vector or a matrix, respectively), which is the conjunction of the formulas $\ker_{Y_1^i}(v_j)$, $j = 1, \ldots, m$ and the following:

$$'\forall u \exists (a_1, \ldots, a_m) \quad \wedge \exists v' \wedge \exists v''$$

$$\ker_{Y_1^{i-1}}(v') \wedge \ker_{Y_1^{i+1}}(v'') \quad \wedge v' + Y_1 v'' + \sum_{i=1}^{m} a_i v_i = u',$$

where a_j are scalars, and v', v'', u are abbreviations for r-vectors (all existential quantifiers in this formula range over the residue field sort). This formula states that the vectors $v_1, \ldots, v_m$ span $\ker(Y_1^i)$ modulo $\ker(Y_1^{i-1}) + Y_1 \ker(Y_1^{i+1})$. Let J be the matrix of the quadratic form q_V, that is, the matrix from the definition of $\mathbf{G}$ as a subgroup of $\mathbf{GL}(r)$. Let '$\mathrm{Gram}(v_1, \ldots, v_c)$' stand for the matrix of formal expressions that gives the Gram matrix of the basis $v_1, \ldots, v_c$ with respect to the quadratic form defined by the matrix ${}^t(Y_1^{i-1} w) J w'$.

If the ϵ_i given is 1, let '$\psi_i(y)$' be the formula:

$$'\exists (v_1, \ldots, v_{c_i}) \ \wedge \ \mathrm{q\text{-}span}(Y_1, v_1, \ldots, v_{c_i})$$

$$\wedge \mathrm{li}(v_1, \ldots, v_{c_i}) \ \wedge \ \exists (z \in k \quad z^2 = \det(\mathrm{Gram}(v_1, \ldots, v_{c_i})) \ \wedge \ z \neq 0).'$$

If the ϵ_i given is 0, let '$\psi_i(y)$' be the formula:

$$'\exists (v_1, \ldots, v_{c_i}) \ \wedge \ \mathrm{q\text{-}span}(Y_1, v_1, \ldots, v_{c_i})$$

$$\wedge \mathrm{li}(v_1, \ldots, v_{c_i}) \ \wedge \ \nexists (z \in k \quad z^2 = \det(\mathrm{Gram}(v_1, \ldots, v_{c_i})) \ \wedge \ z \neq 0).'$$

Finally, it follows from Section 5.7 that the set $W_{C,\lambda}^E$ is defined by the conjunction of $\psi_i(y)$ for all i and the formula defining the set $W_\lambda^E(\Gamma_{\alpha,E})$. □

4.3. **Proof of Theorem 2.** Let $\mathcal{K}$ be the collection of local fields as in the statement of the theorem. Let $E \in \mathcal{K}$. By the formulas (3) – (5), the value $\mathrm{vol}(I_E)\Theta_w(\Gamma_{\alpha,E})$ is a sum over λ of the expressions of the form $\sum_C \rho(C) q^{l_\lambda} \mathrm{vol}(W_{C,\lambda}^E(\Gamma_{\alpha,E}))$. We are now ready to write down the corresponding motivic expression for each of these terms.

Let us fix λ for now. Let $\mathcal{X} = \mathbf{G} \times \mathbf{G}$ as before. By Lemmas 8 and 9, the sets $W_{C,\lambda}^E(\Gamma_{\alpha,E})$ are definable subsets of $\mathcal{X}(E)$, i.e., there exist formulas

$\phi_{C,\lambda}$ in Pas's language, such that $W^E_{C,\lambda}(\Gamma_{\alpha,E}) = Z(\phi_{C,\lambda}, E)$ (in the notation of Appendix, Section 5.2). Let $M_{C,\lambda} = \mu(\phi_{C,\lambda}) \in \hat{K}^{mot}_0(\mathrm{Var}_F)_{\mathbb{Q}}$ be the arithmetic motivic volume of the formula $\phi_{C,\lambda}$ (see Appendix, Section 5.5).

By [37], $\rho(C)$ is a polynomial in q for each C, where q is the cardinality of the residue field of E. Denote this polynomial by $f_C(q)$, and let $\tilde{M}_{C,\lambda} = f_C(\mathbb{L})\mathbb{L}^{l_\lambda} M_{C,\lambda}$, where $\mathbb{L}$ is the Lefschetz motive (see Appendix, Section 5.4), so that $\tilde{M}_{C,\lambda}$ is also an element of the ring $\hat{K}^{mot}_0(\mathrm{Var}_F)_{\mathbb{Q}}$.

Finally, let
$$M_\lambda = \sum_{(\Lambda,(\epsilon_i))} \tilde{M}_{C,\lambda} \in \hat{K}^{mot}_0(\mathrm{Var}_F)_{\mathbb{Q}},$$
where $(\Lambda, (\epsilon_i))$ is the data parametrizing the unipotent conjugacy classes C, as in Section 5.7.

By Theorem 8.3.1, [9] (respectively, Theorem 8.3.2 in the function field case, see also Appendix, Section 5.5),

$$(8) \qquad q^{l_\lambda} \rho(C) \iint_{W^E_{C,\lambda}(\Gamma_{\alpha,E})} 1\, dy d\gamma = TrFrob_E(M_{C,\lambda})$$

for almost all completions E of F, and hence the sum of the expressions on the left-hand side of (8) over C equals $TrFrob_E(M_\lambda)$ for almost all E.

In order to obtain the final result, it remains to sum up both sides of the above formula over λ. In order to be able to do this, we need to take care of the following subtle "uniformity" issue. For each local field E individually, we know that the sum over λ is finite, by Lemma 5. However, we do not know whether this finite set of indices λ (denoted by A_Γ in Lemma 5) is *the same* for almost all places. Another difficulty (which would be automatically resolved if we knew the affirmative answer to the above question) is that for each λ, the formula (8) holds for all but a finite number of places. We need to show that overall (i.e. for all λ altogether) the set of places that needs to be eliminated is finite. These difficulties are taken care of by the following Theorem due to T.C. Hales [18, Theorem 2].

First, we need to introduce a notation. Let $\vartheta(m_1, \ldots, m_l)$ be a formula in Pas's language, with free variables ranging over $\mathbb{N}$. Let $\mathcal{K}$ be a collection of local fields. For $E \in \mathcal{K}$, denote by $\vartheta^E(m_1, \ldots, m_l)$ the interpretation of ϑ in the model for Pas's language given by the field E.

Lemma 10. [18, Theorem 2] *Let $\vartheta(m_1, \ldots, m_l)$ be a formula in Pas's language. Suppose that there exists a finite set of prime numbers $S \subset \mathbb{N}$, such that for any $E \in \mathcal{K}$, if the residual characteristic of E is not in S, then $\{(m_1, \ldots, m_l) \in \mathbb{N}^l \mid \vartheta^E(m_1, \ldots, m_l)\}$ is a bounded subset of $\mathbb{N}^l$. Then there exists a finite set of primes S', $S \subset S' \subset \mathbb{N}$, and exists a bounded set $\mathcal{C} \subset \mathbb{N}^l$, such that for any $E \in \mathcal{K}$, if the residual characteristic of E is not in S', then $\{(m_1, \ldots, m_l) \in \mathbb{N}^l \mid \vartheta^E(m_1, \ldots, m_l)\} \subset \mathcal{C}$.*

For each element of the finite set of parameters corresponding to the conjugacy classes C, we apply Lemma 10 to the formula $\vartheta(\lambda) := `\exists x :$

$\phi_{C,\lambda}(x)$', where $\phi_{C,\lambda}(x)$ is the formula of Lemma 9 that defines the subsets $W_{C,\lambda}^E(\Gamma_{\alpha,E})$. That is, $\vartheta^E(\lambda)$ takes the value 'true' iff the set $W_{C,\lambda}^E(\Gamma_{\alpha,E})$ is not empty (here we understand the symbol λ as an abbreviation for the l-tuple of variables $(m_1, \ldots, m_l)$.) By Lemma 5, for any E such that the notation $W_{C,\lambda}^E$ makes sense (that is, for all but finitely many fields $E \in \mathcal{K}$), the set $\{\lambda \mid \vartheta^E(\lambda)\}$ is finite. Hence, the assumption of Lemma 10 holds. We conclude that there exists a finite set of primes S' such that whenever the residual characteristic of E does not fall in this set, $\{\lambda \mid \vartheta^E(\lambda)\} \subset \mathcal{C}$, where $\mathcal{C}$ is a bounded (i.e., finite) subset of $\mathbb{N}^l$.

Finally, set $M_{\alpha,w} = \sum_\lambda M_\lambda$, where the summation is over the union of finite sets $\mathcal{C}$ that correspond to the data defining the conjugacy classes C (which is a finite union of finite sets, so the sum is finite). This completes the proof.

5. Appendix: Some Background

This section is a compilation of brief reviews of various concepts and techniques from logic, algebraic geometry, and representation theory that we used above.

5.1. **Pas's language.** We need to deal with families of local fields (e.g. all possible nonarchimedean completions of a given number field). Therefore, it is necessary to set up a framework that allows to exploit the structure of a local field without referring to its individual features such as the uniformizer of the valuation, for example. This is achieved by using a formal language of logic, which has the terms for the valuation, etc., but does not have the term for a uniformizer. This language is called Pas's language.

A sentence in Pas's language for a valued field E is allowed to have variables of three kinds: variables running over the valued field E (the *valued field sort*), variables running over its residue field k (the *residue field sort*), and variables running over $\mathbb{Z}$ (the *value sort*). Formally, it is a three-sorted first order language [10].

For variables of the valued field sort and for variables of the residue field sort, the language has the operations of addition ('+') and multiplication ('×'); for variables of the value sort (i.e., for $\mathbb{Z}$), only addition is allowed. The value sort, additionally, has symbols $\leq$, and $\equiv_n$ for congruence modulo each value $n \in \mathbb{N}$.

The language also has symbols for universal ('∀') and existential ('∃') quantifiers, and standard symbols $\wedge$, $\vee$, $\neg$, respectively, for logical conjunction, disjunction, and negation. We note that the restriction of Pas's language to the residue field sort coincides with the *first order language of rings* [10].

Naturally, there are symbols denoting all the integers in the value sort. In the valued field sort, there are symbols '0' and '1', defined as the symbols denoting the additive and multiplicative unit, respectively. Once we have these, we can formally add to the language the symbols denoting other

integers in the valued field sort. Thus, '2' is the abbreviation for '$1 + 1$', '-1' is the abbreviation for '$\exists x, x + 1 = 0$', etc. Notice that there are no symbols denoting other elements of the valued field, in particular, there is no symbol for the uniformizer. This, of course, agrees with our goal of being able to use the same language for any local field within a given family. We illustrate the allowed operations with the following example.

Example 11. We are not allowed to use any symbols for *constants* in the valued field. For example, the expression '$a_1 x + a_2 y = 0$' makes sense in Pas's language *only* if a_1, a_2, x and y are all treated on equal footing, as *variables* of the same sort.

The language also contains the following symbols denoting functions: the symbol 'ord' for the valuation – a function from the valued field sort to the value sort, and the symbol '$\overline{\mathrm{ac}}$' to denote the *angular component* map from the valued field sort to the residue field sort (the role of this map will be explained in Section 5.2). We also add the symbol '∞' to the value sort, to denote the valuation of 0.

There is a theorem due to Pas [29] that this language admits quantifier elimination of the quantifiers ranging over the valued field sort and over the value sort (this is the reason for the absence of multiplication for the integers: otherwise, by Gödel's theorem, there would be no quantifier elimination). This means that every formula in Pas's language can be replaced by an equivalent formula without quantifiers ranging over the valued field sort or over the value sort. Further, a theorem due to Presburger [30] states that the quantifiers ranging over the residue field can also be eliminated, so that every formula in Pas's language can be replaced by an equivalent formula without quantifiers. Ultimately, it is this property that makes *arithmetic motivic integration* possible [9].

5.2. **Definable subsets for p-adics.** This subsection is, essentially, quoted from [9, Sections 8.2, 8.3].

Let E be the field of fractions of a complete discrete valuation ring $\mathcal{O}_E$ with finite residue field k. It is possible to think of E as a *structure* (in the sense of logic) for Pas's language. That is, we can let the variables in the formulas in Pas's language range over E and k respectively, and then each formula will have true/false value. In order to match Pas's language with the structure of the field E completely, we need to give a meaning to the symbols that express functions in Pas's language. We fix a uniformizing parameter ϖ. The valuation on E is normalized so that $\mathrm{ord}(\varpi) = 1$. If $x \in \mathcal{O}_E^*$ is a unit, there is a natural definition of $\overline{\mathrm{ac}}(x)$ – it is the reduction of x modulo the ideal (ϖ). Define, for $x \neq 0$ in E, $\overline{\mathrm{ac}}(x) = \overline{\mathrm{ac}}(\varpi^{-\mathrm{ord}(x)} x)$, and $\overline{\mathrm{ac}}(0) = |0| = 0$.

Now let us take F – a finite extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$. Following [9], we let $R = \mathcal{O}[\frac{1}{N}]$, for some non zero integer N which is a multiple of the discriminant of F. Let v be a closed point of $\mathrm{Spec}\, R$. We

denote by F_v the completion of the localization of R at v, by $\mathcal{O}_{F_v}$ – its ring of integers, and by k_v – the residue field. If N equals the discriminant of F, then $\{F_v\}$ is the family of all completions of F at the places lying over the primes in $\mathbb{Q}$ that do not ramify in F.

In Section 5.1, we formally added to Pas's language the symbols to denote all integers in the valued field sort. In order to work with the family of fields $\mathcal{K} = \{F_v\}$, it is convenient to also add to Pas's language, for every element of the global field F, a symbol to denote this element in the valued field sort. This allows to "have constants from F" in the formulas in Pas's language. We will call this extended language the Pas's language for the valued field $F((t))$. The reason for this is that $F((t))$ is, naturally, a valued field (t being the uniformizer of the valuation), and every logical formula in the extended Pas's language for this field also makes sense as a formula in the extended Pas's language for every completion of the field F. It is through interpreting the Pas's language formulas in the model given by the field $F((t))$ that one eventually arrives at geometric objects associated with them (see [17, Section 6.1]).

Definition 12. Let ϕ be a formula in the Pas's language for the valued field $F((t))$, with m free variables running over the valued field sort and no free variables running over the residue field sort or the value sort. For each v – a place of F, and $E = F_v$ – the corresponding completion, denote by $Z(\phi, \mathcal{O}_E)$ the subset of $\mathcal{O}_E^m$ defined by the formula ϕ: $Z(\phi, \mathcal{O}_E) = \{(x_1, \ldots, x_m) \in \mathcal{O}^m \mid \phi(x_1, \ldots, x_m)\}$ (recall that ϕ has the values true/false). A subset $B \subset \mathcal{O}_E^m$ is called *definable* if there exists a formula ϕ such that $B = Z(\phi, \mathcal{O}_E)$.

For example, if $V \hookrightarrow \mathbb{A}^m$ is an affine variety, $V(\mathbb{Z}_p)$ is a definable subset of $\mathbb{Z}_p^m$ (it can be defined by a logical formula with m free variables which does not use the symbols '$\overline{\mathrm{ac}}$' and 'ord': just the polynomial relations defining the variety V).

So far, we have defined the notion of a definable subset of $\mathcal{O}_E^m = \mathbb{A}^m(\mathcal{O}_E)$. This definition extends naturally to give a notion of a *definable* subset of an affine variety $\mathcal{X}$ over $\mathcal{O}_E$.

Let $\mathcal{X}$ be a smooth variety over F of dimension d. There is a natural d-dimensional measure on $\mathcal{X}(\mathcal{O}_{F_v})$ for each v, which we shall denote by $vol_{\mathcal{X}, F_v}$. This is the Serre-Oesterlé measure, [28], [34]. All definable subsets of $\mathcal{X}(\mathcal{O}_{F_v})$ are measurable with respect to this measure, when F has characteristic 0, [5]. This measure is defined by requiring that the fibers of the reduction map modulo ϖ_v have volume q^{-d}, where q is the cardinality of the residue field $\mathcal{O}_{F_v}/(\varpi_v)$.

In the Section 5.5, we describe the theory of *arithmetic motivic integration* which associates algebraic geometric objects with logical formulas. For each place v, there is a linear operator (Frobenius) acting on the cohomology of these objects, and its trace on the object associated with the formula ϕ equals the Serre-Oesterlé volume of the set $Z(\phi, \mathcal{O}_{F_v})$ for almost all v.

5.3. **Motivic measures.** The purpose of this subsection is twofold. The first goal is to outline the context for the ideas from the theory of motivic integration that we are using and to give some background references. The second objective is to let someone who is an expert in motivic integration know exactly to what extent the theory is being used in this paper. All the precise definitions and statements that we need are quoted in the next three subsections.

The term *motivic integration* first appeared in M. Kontsevich's lecture at Orsay in 1995. Let k be an algebraically closed field of characteristic 0, and let $\mathcal{X}$ be an algebraic variety over k. *Motivic measure* lives on the *arc space* of $\mathcal{X}$, i.e. on the "infinite-dimensional variety" whose set of k-points coincides with the set of $k[[t]]$-points of $\mathcal{X}$. It is a measure in every sense (i.e., it is additive, it transforms under morphisms in a such a way that makes it analogous to measures on real or p-adic manifolds, etc.), but its nature is algebraic, and its values are not real or complex numbers, but, roughly speaking, equivalence classes of varieties. The theory of motivic integration is, in fact, a theory about the motivic measure. The functions [on the arc space] that can be integrated against this measure are scarce. Due to algebraic nature of the measure, the algebra of measurable subsets of the arc space is rather coarse: it is, essentially, generated by the sets "growing out" of subvarieties of $\mathcal{X}$. For background information on the original motivic integration we refer to [3], [7], [8], and to the Bourbaki talk by E. Looijenga [25].

The *arithmetic motivic integration* developed by J. Denef and F. Loeser in 1999 [9] uses similar ideas, but it is an independent theory. The main difference is that it is adapted to deal with sets of rational points of the given variety over various fields, as opposed to the less flexible original theory which is suited only to deal with the arc space of the given variety, and not its rational points. The *arithmetic motivic volume* takes values in *virtual Chow motives.*

Instead of the arc space itself, the arithmetic motivic volume lives on *definable subassignments* of the functor of its points. This distinction allows one to work directly at the level of rational points [9]. The *definable subassignments* are, in a sense, a geometric incarnation of logical formulas in Pas's language. We will not need these geometric objects, since all the varieties we deal with here are smooth and affine. Hence, we will assign "motivic volumes" to logical formulas directly.

5.4. **The ring $\hat{K}_0^{mot}(\mathbf{Var}_F)_{\mathbb{Q}}$.** Here we define the ring of values of the arithmetic motivic volume. The *volume* itself will be defined in the next subsection. This section is, essentially, quoted from [9, Section 1.3], with one modification: the definition of the ring of values was greatly simplified by Denef and Loeser in [6]. Hence, we are using the version of this ring that appears in [6] rather than the original one.

Let F be a field. Let us denote by Mot_F the category of Chow motives over F, with coefficients in $\mathbb{Q}$. The objects in Mot_F are, formally, equivalence classes of triples (S, p, n), where S is a proper and smooth scheme over F, p is an idempotent correspondence on S with coefficients on $\mathbb{Q}$ [that is, an algebraic cycle in $S \times S$ such that $p^2 \simeq p$, where p^2 is the product of p with itself], and $n \in \mathbb{Z}$. Some details about Chow motives can be found in [32].

Even though Chow motives do not form an abelian category, they are sufficiently suited for our purposes, because the category Mot_F is pseudo-abelian, and therefore the notion of its Grothendieck group makes sense. We denote this Grothendieck group by $K_0(\mathrm{Mot}_F)$. This is the additive group of equivalence classes of formal linear combinations of objects of Mot_F (with the natural notion of equivalence so that if $M = A \oplus B$ then $[M] = [A] + [B]$). Since the category of Chow motives has a tensor product, $K_0(\mathrm{Mot}_F)$ can be given the structure of a ring. There is a natural inclusion of the set of objects of the category Mot_F into its Grothendieck ring (each object can be identified with its own equivalence class in $K_0(\mathrm{Mot}_F)$).

Let Var_F be the category of algebraic varieties over F. For a *smooth projective* variety S, there is a Chow motive canonically associated with it – the triple $(S, id, 0)$. There exists a unique embedding

$$\chi_c : \mathrm{Var}_F \to \mathrm{Mot}_F$$

such that for smooth projective S, $\chi_c(S) = (S, id, 0)$.
Let $\mathbb{L} = (\mathrm{Spec}\, F, id, -1)$ be the Lefschetz motive (see [32] for details). It is the image, under the morphism χ_c, of the affine line $\mathbb{A}^1$ (note that the affine line is not a projective variety, and this is the reason the integer component of the Chow motive associated with it is not 0). The map χ_c induces the morphism of Grothendieck rings [11], [13]:

$$\chi_c : K_0(\mathrm{Var}_F) \to K_0(\mathrm{Mot}_F).$$

The object $\mathbb{L}$ is invertible (with respect to tensor product) in the category of Chow motives. Hence, the map χ_c extends to the localization of $K_0(\mathrm{Var}_F)$ at $[\mathbb{A}^1]$. We denote by $K_0^{mot}(\mathrm{Var}_F)$ the image of this extended map χ_c. Finally, set $K_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}} = K_0^{mot}(\mathrm{Var}_F) \otimes \mathbb{Q}$.

For more details of the definition of the ring of values of the arithmetic motivic volume we refer to [9, Section 1.3] and [6, Section 6.3]. In the next subsection, we describe the *arithmetic motivic volume*, which is a function on logical formulas [rather, on *definable subassignments* as in [9]] taking values, roughly, in this ring. More precisely, it is necessary to complete the ring $K_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$ with respect to a certain *dimensional filtration* in order to have a meaningful "measure theory" with values in it. To define the filtration of the ring of virtual Chow motives, we first define a filtration on $K_0(\mathrm{Var}_F)_{loc}$ (where the subscript '*loc*' stands for localization at $[\mathbb{A}^1]$). For $m \in \mathbb{Z}$, let $F^m K_0(\mathrm{Var}_F)_{loc}$ be the subgroup of $K_0(\mathrm{Var}_F)$ generated by the elements of the form $[S]\mathbb{L}^{-i}$ with $i - \dim S \geq m$. This defines a decreasing filtration F^m. The image of this filtration under the map χ_c is a decreasing

filtration on $K_0^{mot}(\mathrm{Var}_F)$, which naturally induces a filtration on the full ring $K_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$. The completion of this ring with respect to this filtration is denoted by $\hat{K}_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$.

Remark 13. It is important to note that the volumes of all the objects that we will be dealing with are contained in the image of the ring $K_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$ in $\hat{K}_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$ under the completion map, if we adjoin to it all the elements of the form $[\mathbb{L}^i - 1]^{-1}$ for positive integers i (see [9, Remark 8.1.2]). In fact, there is now new, yet unpublished, version of the theory of motivic integration (due to R. Cluckers and F. Loeser), which does not use the completion at all. This new theory allows a refinement of our results, namely, the virtual Chow motives responsible for the character values should automatically lie in the ring $K_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}[\mathbb{L}^i - 1]_{i \geq 1}$, rather than in a less understood complete ring $\hat{K}_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$.

If v is a place of F, there is an action of $Frob_v$ on the elements of $\hat{K}_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$ that comes from the Frobenius action on the Chow motives. The trace of the Frobenius operator on a Chow motive is the alternating sum of the traces of Frobenius acting on its l-adic cohomology groups, and it is an element of $\bar{\mathbb{Q}}_l$ (where l is a prime number), see [9, Section 3.3]. In all the cases that we consider (following Denef and Loeser), this number turns out to lie in $\mathbb{Q}$ (in particular, the choice of l doesn't matter). It is the trace of Frobenius action that allows to relate the values of the motivic measure, that are elements of $\hat{K}_0^{mot}(\mathrm{Var}_F)_{\mathbb{Q}}$, with the usual p-adic volumes, that are rational numbers.

5.5. Arithmetic motivic integration.
A beautiful very short description of the theory of arithmetic motivic integration can be found in [17, Section 6.1].

Let $\mathcal{X}$ be a smooth d-dimensional affine variety over F, where F is a global field as above. Let ϕ be a logical formula in Pas's language for the field $F((t))$ with m free variables ranging over the valued field sort and no bound variables over the value sort. There is a natural notion of the formula ϕ defining a subset of $\mathcal{X}$: suppose that $\mathcal{X}$ is embedded in $\mathbb{A}^m$ in some way. Then we can assume that $\phi(x_1, \ldots, x_m)$ is true only if $x_1, \ldots, x_m$ satisfy the polynomial equations defining $\mathcal{X}$. In other words, this happens when $Z(\phi, \mathcal{O}_{F_v}) \subset \mathcal{X}(F_v)$ for almost all places v of F, see [9, Proposition 5.2.1 and Section 8.3].

In [6, Section 6.3] (following the original method of [9, Section 3]), Denef and Loeser associate elements of the ring $K_0^{mot}(Var_F)_{\mathbb{Q}}$ to such formulas ϕ. This mapping is additive in a natural sense (see [9, Proposition 3.4.4]). This map is unique; it is denoted by $\chi_c(\phi)$ in [9], but we will denote it by $\mu(\phi)$. This map eventually extends to the formulas that involve quantifiers. In order to define this extension, the authors use the language of definable subassignments of the functor of points of $\mathfrak{L}(\mathbb{A}^m)$. The resulting map on subassignments is called *arithmetic motivic volume*, see [9, Section 6]. We

will not need the precise definition of the arithmetic motivic volume or the language of subassignments. We only need its existence, finite additivity, and the following two theorems. In the form that we state these theorems, they are special cases of Theorems 8.3.1 and 8.3.2 [9], respectively, corresponding to the case when the variety $\mathcal{X}$ is smooth and affine, and the definable subassignment is defined by just one formula on the whole space, in the terms of [9].

Theorem 14. [9, Theorem 8.3.1] *Let F be a finite extension of $\mathbb{Q}$ with the ring of integers $\mathcal{O}$, and $R = \mathcal{O}[\frac{1}{N}]$, for some non zero integer N. Let $\mathcal{X}$ be an affine variety over R, and let ϕ be a logical formula as above. Then there exists a non zero multiple N' of N, such that, for every closed point v of $\operatorname{Spec}\mathcal{O}[\frac{1}{N'}]$,*

$$Tr\,Frob_v\big(\mu(\phi)\big) = vol_{\mathcal{X},F_v}(Z(\phi, \mathcal{O}_{F_v})).$$

In the case when the local fields under consideration are of finite characteristic, there is a slightly stronger result:

Theorem 15. [9, Theorem 8.3.2] *Let F be a field of characteristic 0 which is the field of fractions of a normal domain R of finite type over $\mathbb{Z}$. Let $\mathcal{X}$ be a variety over R, and let ϕ be a logical formula. Then there exists a nonzero element f of R, such that, for every closed point x of $\operatorname{Spec} R_f$, $Z(\phi, \mathbb{F}_x[[t]])$ is $vol_{\mathbb{F}_x[[t]]}$-measurable and*

$$Tr\,Frob_x\big(\mu(\phi)\big) = vol_{\mathcal{X},F_x[[t]]}(Z(\phi, \mathbb{F}_x[[t]])).$$

This completes our survey of arithmetic motivic integration. The next two subsections are devoted to a summary of the facts from representation theory that we are using.

5.6. Deligne-Lusztig representations of classical groups.
Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\bar{\mathbb{F}}$, where $\mathbb{F} = \mathbb{F}_q$ is a finite field, $q = p^m$. Deligne and Lusztig [4] constructed a large class of representations of groups of the form $\mathbf{G}^{\mathbb{F}}$ in vector spaces over $\bar{\mathbb{Q}}_l$ with $l \neq p$. By $\mathbf{G}^{\mathbb{F}}$ we denote the group of fixed points of $\mathbf{G}(\bar{\mathbb{F}})$ under the Frobenius map associated with q.

These representations are parametrized by pairs (T, χ) where T is a maximal torus in $\mathbf{G}^{\mathbb{F}}$ defined over $\mathbb{F}$ and χ is an irreducible character of T. A representation corresponding to the pair (T, χ) is denoted by $R_{T,\chi}^{\mathbf{G}^{\mathbb{F}}}$.

Definition 16. An irreducible representation of $\mathbf{G}^{\mathbb{F}}$ is called *Deligne-Lusztig* if it is equivalent to $R_{T,\chi}^{\mathbf{G}^{\mathbb{F}}}$ for some (T, χ).

In this paper we deal with representations of p-adic groups that are obtained from Deligne-Lusztig representations by a certain natural procedure described in Section 2.2. The reason we restrict our attention only to Deligne-Lusztig representations is the fact that the values of their characters at unipotent elements in $\mathbf{G}^{\mathbb{F}}$ are given by polynomials in q. We use this fact in an essential way in the proof of our main result.

More specifically, if u is a unipotent element of $\mathbf{G}^{\mathbb{F}}$, the value of the character of $R_{T,\chi}^{\mathbf{G}^{\mathbb{F}}}$ at u can be expressed as a sum of values of Green functions $Q_T^{\mathbf{G}}(u)$ (see, e.g., [36, Theorem 6.8]) (in particular, it does not depend on χ). It was proved by B. Srinivasan in [37] that the values of $Q_T^{\mathbf{G}}(u)$ are polynomials in q (the polynomial itself depends on the conjugacy class of u) if $\mathbf{G}$ is symplectic or odd special orthogonal, and the characteristic p is odd (since everything we do here is only up to a finite number of primes, this is not a restriction).

5.7. Parametrization of unipotent conjugacy classes.
Here we review the parametrization of nilpotent orbits in the classical p-adic Lie algebras given by Waldspurger [38]. We quote it from [38, Sections I.5, I.6]. It is used in an essential way in the proof of Lemma 9.

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sp}(r)$ or $\mathfrak{so}(r)$ with r odd, and let $X \in \mathfrak{g}$ be a nilpotent element. Denote by (V, q_V) the underlying vector space of $\mathfrak{g}$ with the quadratic form from the definition of $\mathfrak{g}$ on it.

Consider the set of all partitions $\Lambda = (\Lambda_j)$ of r with the following properties:

- In the symplectic case, for all odd $i \geq 1$, $c_i(\Lambda)$ is even
- In the orthogonal case, for any even $i \geq 2$, $c_i(\Lambda)$ is even

(here $c_i(\Lambda)$ is the number of Λ_j's that equal i).

We can associate with X a partition Λ of r: for all integers $i \geq 1$, $c_i(\Lambda)$ is the number of Jordan blocks of X of length i in the natural matrix representation. This partition automatically satisfies the above conditions.

For all $i \geq 1$, set $V_i = \ker(X^i)/[\ker(X^{i-1}) + X\ker(X^{i+1})]$. In the symplectic (resp, orthogonal) case, define the quadratic form $\tilde{q}_i$ on $\ker(X^i)$, for all even i (resp., odd), by:

$$\tilde{q}_i(v, v') = (-1)^{\left[\frac{i-1}{2}\right]} q_V(X^{i-1}(v), v').$$

Passing to a quotient, we get a non-degenerate form q_i on V_i.

In the orthogonal case, the forms q_i satisfy the condition

$$\bigoplus_{i \text{ odd}} q_i \sim_a q_V,$$

where $\sim_a$ indicates that the two forms have the same anisotropic kernel.

The correspondence described above is a bijection between the set of nilpotent orbits (under conjugation by elements of G) in $\mathfrak{g}$ and the set of pairs $(\Lambda, (q_i))$ with Λ – a partition of r and (q_i) – a collection of quadratic forms of dimensions $c_i(\Lambda)$ satisfying the conditions mentioned above.

The same classification holds for finite fields. We also observe that over a finite field $\mathbb{F}_q$, an equivalence class of a quadratic form is determined by its rank and discriminant in $\mathbb{F}_q^*/\mathbb{F}_q^{*2}$ [33]. The nilpotent orbits in the Lie algebra are in bijection with the unipotent conjugacy classes in the group, when the characteristic p is large enough [20]. Explicitly, the bijection is given by Cayley transform $x \mapsto (1-x)(1+x)^{-1}$.

Let $\mathbf{G}$ be a simply connected algebraic group of symplectic or odd orthogonal type.

Finally, we see that the unipotent conjugacy classes of $\mathbf{G}^{\mathbb{F}_q}$ are in bijection with the following data (cf. also [35, I 2.9] or [26, 11.1]):

- a partition Λ of r with certain properties;
- a representative (one of the two possible choices) of $\mathbb{F}_q^*/\mathbb{F}_q^{*2}$ for each even (resp. odd) $c_i(\Lambda)$.

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Institute for Advanced Study, 1 Einstein Dr., Princeton NJ 08540, USA
E-mail address: julygord@math.ias.edu